Solution of Inverse Problem of Transonic Flow on S_2 Surface Using an Elliptic Algorithm

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The flow on an S_2 surface with given $V_\theta r$ variation has been widely used in practice for direct and inverse problems of three-dimensional flows in turbomachines. The fact that the differential equations governing the flow remain elliptic as long as the meridional component of the absolute velocity is lower than the speed of sound, even in cases where the flow relative to the rotating blade is supersonic, is rigorously proved on the basis of the original system of first-order differential equations, as well as of the single second-order principal equation expressed in terms of the stream function. This provides a sound mathematical basis for solving the difficult transonic S_2 surface flow, especially for the design problem, by a well-developed method for solving an elliptic equation. It is pointed out, however, that in so doing, it is important that the prescribed variation of $V_\theta r$ should have an appropriate abrupt change at the shock so that an accurate detailed flow variation throughout the rotor passage can be obtained. The preceding argument is illustrated by an example in which the transonic $S_{2,m}$ flow in a research axial-flow transonic compressor rotor is obtained by the use of a conventional method for solving the elliptic second-order partial differential equation for the stream function.

Nomenclature

a	= speed of sound
c_p	= specific heat at constant pressure
D/Dt	= differentiation with respect to time following relative motion of fluid particle
F	= force per unit mass of gas
H	= absolute stagnation enthalpy, $h + V^2/2$
h	= enthalpy per unit mass of gas, $u + p/\rho$
I	= relative stagnation rothalpy
M	= Mach number
M_{1n}	= normal component of Mach number in front of shock
$\boldsymbol{n}$	= unit vector norm to stream surface
p	= gas pressure
R	= gas constant
$\boldsymbol{r}$	= radius vector from the machine axes
r, θ, z	= absolute cylindrical coordinates
r, φ, z	= relative cylindrical coordinates
s	= entropy per unit mass of gas
T	= absolute temperature, K
U	= velocity of blade element at radius r
u	= internal energy per unit mass of gas
$\boldsymbol{V}, \boldsymbol{W}$	= absolute and relative velocities of gas, respectively
W_1	= meridional component of $\boldsymbol{W}$
W_t	= projection of $\boldsymbol{W}$ to a direction tangent to shock surface
x^1, x^2	= general nonorthogonal curvilinear coordinates in meridional plane
W^i	= physical component of $\boldsymbol{W}$ along coordinate x^i
α	= Mach angle, $\sin^{-1}(1/M)$
β	= angle between span of swept wing and incident flow
θ_{12}	= angle between x^1 and x^2 line
κ	= ratio of specific heats
λ	= slope of characteristic line
ρ	= density of gas
τ	= circumferential thickness of S_2 stream filament
ψ	= stream function defined on stream surface S_2
ω	= angular velocity of rotor

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Introduction

A GENERAL theory of steady three-dimensional flow of a nonviscous fluid in subsonic and supersonic turbomachines is presented in Ref. 1. The solution of the three-dimensional direct (analysis) or inverse (design) problem is obtained by calculating an appropriate combination of flows on relative stream surfaces whose intersection with a z plane somewhere in the flowfield forms, respectively, a circular arc (S_1 family) and a radial or nearly radial line (S_2 family). The basic equations of fluid mechanics and thermodynamics governing the flow are combined to form a principal equation that is expressed in terms of a stream function defined on the stream surface and is a quasi-linear second-order partial differential equation. The character of this equation, i.e., whether it is of the elliptic or hyperbolic type, depends on the magnitude of the velocity with respect to the local speed of sound. In the case of the inverse (design) problem, where a certain desirable variation of the angular momentum of the fluid $V_\theta r$ is prescribed on a mean S_2 surface $S_{2,m}$, which is about midway (based on mass flow) between two adjacent blades, the principal equation is seen to be hyperbolic or elliptic when the meridional velocity $W_1 \gtrless$ the local speed of sound. Although it appeared to be contradictory to the common belief that the criterion should be based on the relative velocity rather than its meridional component, this equation has been widely used by turbomachine designers, and many transonic turbomachines have been designed on the basis of elliptic calculation of the transonic flow on $S_{2,m}$ (Ref. 2), or its variants, such as the axisymmetric flowfield analysis of Refs. 3 and 4 and the streamline analysis program of Ref. 5. However, two questions remain: one is the validity of the determination of the type of the various principal equations, and the other is

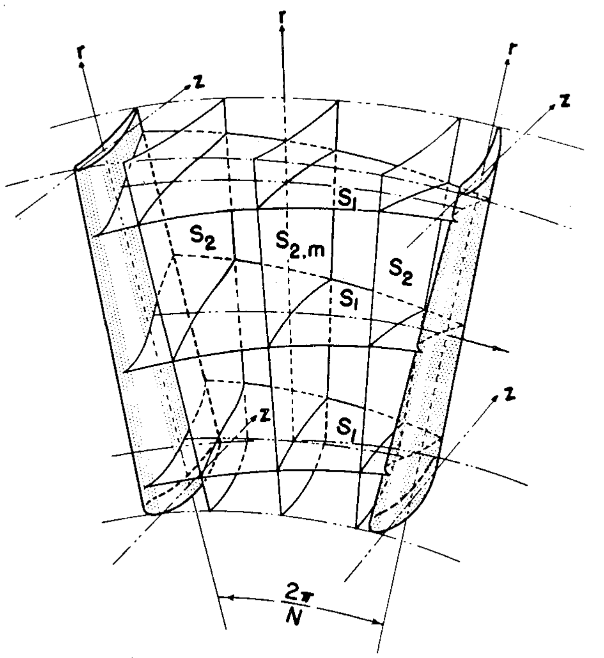

Fig. 1 Relative stream surfaces S_1 and S_2.

the accurate prescription of the design parameter (e.g., $V_\theta r$) and the accurate solution of the principal equation.

In this paper the question about the type of differential equations governing the flow on a stream surface is examined in a more general and complete manner than it was in Ref. 1. Because the theory of characteristics is "most transparent"[6] for systems of first-order differential equations, the present problem is first investigated on the basis of the original first-order differential equations governing the fluid flow on the S_2 stream surface. Then it is investigated on the basis of the second-order principal equation, which is expressed in terms of the stream function. The results obtained both ways are compared. Finally, the importance of an appropriate jump of the prescribed design parameter across the shock is emphasized in order to obtain an accurate solution of the transonic flow on the stream surface. Examples illustrating the difference in the solutions corresponding to the incorrect and correct prescriptions of the design parameter across the shock are given.

Basic Aerothermodynamic Relations

The three-dimensional flow of a nonviscous, compressible fluid through a turbomachine is governed by the following set of basic laws of fluid mechanics and thermodynamics:

For steady relative flow, the continuity equation is

$$\nabla \cdot (\rho \boldsymbol{W}) = 0 \tag{1}$$

For blades rotating at a constant angular velocity, Newton's second law of motion can be expressed by[1]

$$-\frac{1}{\rho}\nabla p = \frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t} = \frac{\mathrm{D}'\boldsymbol{W}}{\mathrm{D}t} - \omega^2 \boldsymbol{r} + 2\boldsymbol{\omega}\times\boldsymbol{W} \tag{2}$$

where $\mathrm{D}'\boldsymbol{W}/\mathrm{D}t$ denotes the relative acceleration. For adiabatic flow, the energy equation derived from the first law of thermodynamics is

$$\frac{\mathrm{D}u}{\mathrm{D}t} + p\frac{\mathrm{D}(\rho^{-1})}{\mathrm{D}t} = 0 \tag{3}$$

From the second law of thermodynamics, a thermodynamic property entropy is defined and is related to T, μ, p, and ρ by

$$T\mathrm{d}s = \mathrm{d}u + p\,\mathrm{d}(\rho^{-1}) \tag{4}$$

When the fluid is a perfect gas, the following relations hold:

$$p = R\rho T \tag{5}$$

$$a^2 = \kappa RT \tag{6}$$

$$h \equiv u + p/\rho = u + RT \tag{7}$$

$$\mathrm{d}h = c_p\,\mathrm{d}T \tag{8}$$

In the investigation of turbomachine flow, it has been found that the use of entropy s and relative stagnation rothalpy[7] I, instead of the usual p and ρ, as the two primary thermodynamic properties of gas, greatly facilitates the solution. The definition of I is[1]

$$I \equiv (h - \omega^2 r^2/2) + W^2/2 \tag{9}$$

Then, from Eqs. (2) to (5), (8), and (9), there are obtained

$$\boldsymbol{W}\times(\nabla\times\boldsymbol{V}) = \nabla I - T\nabla s \tag{10}$$

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0 \tag{11}$$

$$\frac{\mathrm{D}I}{\mathrm{D}t} = 0 \tag{12}$$

It is seen from Eqs. (11) and (12) that I and s remain constant along a relative streamline.

Determining the Type of the System of First-Order Equations Governing S_2 Flow

In Ref. 1, the solution of the three-dimensional flow in a turbomachine is based on the use of two families of relative stream surfaces S_1 and S_2 (see Fig. 1). In the direct problem, the solution is obtained through the successive use of S_1 and S_2 families of stream surfaces, and the shape of the surface and the variation of the stream filament thickness of one family are taken from the solutions obtained for the flow on surfaces of the other family. In the inverse problem, calculation usually starts from an $S_{2,m}$ surface, which is about in the middle of the flow channel formed by two adjacent blades, and on which the variation of a certain flow variable, usually the angular momentum of the fluid $V_\theta r$, and that of τ, which corresponds to a desired blade thickness distribution, are prescribed by the designer. In the following, the type of differential equations governing the direct and inverse problems of fluid flow on S_2 surfaces will be analyzed.

Any variable q on the S_2 surface is considered a function of r and z, i.e.,

$$q = q[r, z, \varphi(r,z)]$$

The change of q on the S_2 surface with respect to a small change in r, while z is held constant (see Fig. 2), is written

$$\mathrm{d}q = \frac{\partial q}{\partial r}\mathrm{d}r + \frac{\partial q}{\partial \varphi}\frac{\partial \varphi}{\partial r}\mathrm{d}r = \left(\frac{\partial q}{\partial r} - \frac{n_r}{n_\varphi r}\frac{\partial q}{\partial \varphi}\right)\mathrm{d}r$$

Thus, the partial derivative following motion on the S_2 surface is written[1]

$$\frac{\bar{\partial} q}{\partial r} = \frac{\partial q}{\partial r} - \frac{\mu}{r}\frac{\partial q}{\partial \varphi} \tag{13}$$

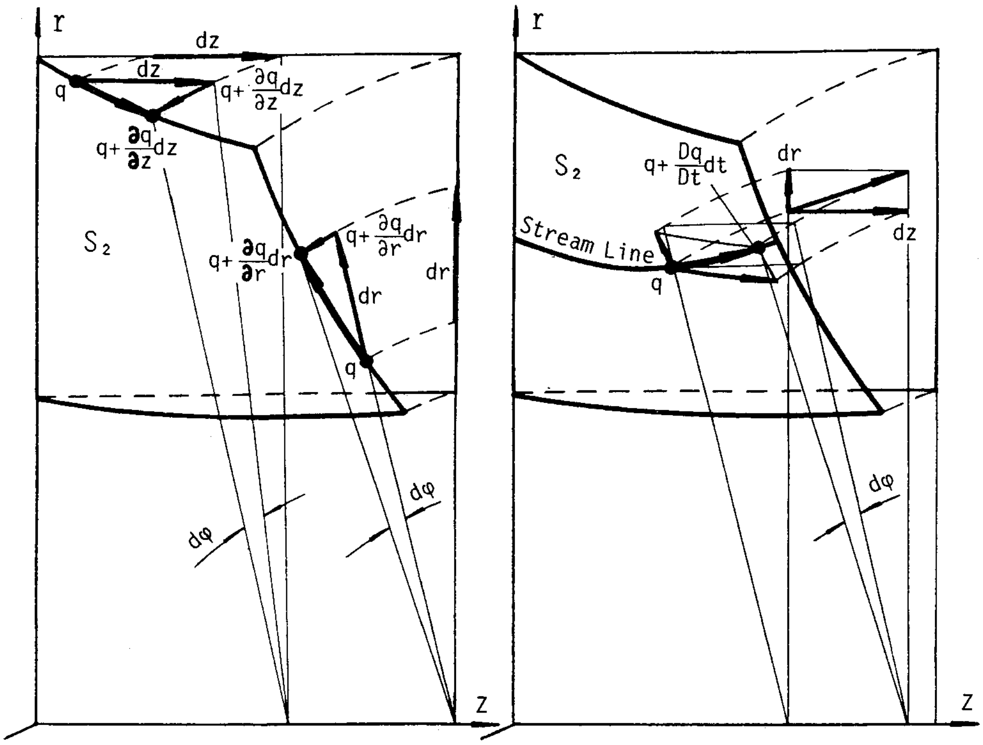

Fig. 2 Partial and total derivatives following motion on S_2.

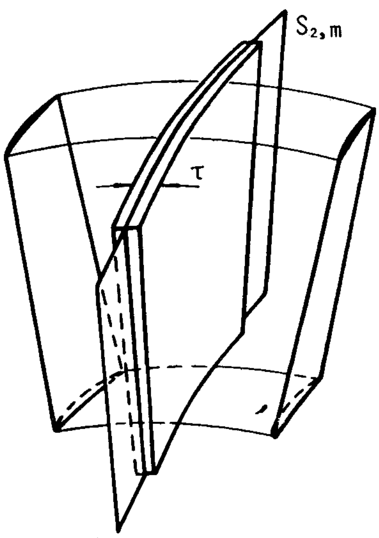

Fig. 3 S_2 stream filament.

where μ denotes n_r/n_φ. Similarly,

$$\frac{\bar\partial q}{\partial z}=\frac{\partial q}{\partial z}-\frac{\nu}{r}\frac{\partial q}{\partial\varphi} \tag{14}$$

where ν denotes n_z/n_φ. The total derivative following motion on $S_{2,m}$ is

$$\frac{Dq}{Dt}=W_r\frac{\bar\partial q}{\partial r}+W_z\frac{\bar\partial q}{\partial z} \tag{15}$$

With the use of the preceding relations, the continuity equation (1) can be written as[1]

$$\frac{\bar\partial(\tau\rho W_r)}{\partial r}+\frac{\bar\partial(\tau\rho W_z)}{\partial z}=0 \tag{16}$$

where τ is proportional to the product of r and B, which is the integrating factor, and can be interpreted as the circumferential thickness of an S_2 stream filament whose central surface is the $S_{2,m}$ surface considered (see Fig. 3).[1] By the use of Eqs. (5–9), Eq. (16) can be put into the following form:

$$\left(1-\frac{W_r^2}{a^2}\right)\frac{\bar\partial W_r}{\partial r}-\frac{W_rW_\varphi}{a^2}\frac{\bar\partial W_\varphi}{\partial r}-\frac{W_rW_z}{a^2}\frac{\bar\partial W_z}{\partial r}-\frac{W_zW_r}{a^2}\frac{\bar\partial W_r}{\partial z}-\frac{W_zW_\varphi}{a^2}\frac{\bar\partial W_\varphi}{\partial z}+\left(1-\frac{W_z^2}{a^2}\right)\frac{\bar\partial W_z}{\partial z}+\frac{\omega^2 rW_r}{a^2}+W_r\frac{\bar\partial \ln\tau}{\partial r}+W_z\frac{\bar\partial \ln\tau}{\partial z}=0 \tag{17}$$

The equation of motion (10) in the radial, tangential, and axial directions become:[1]

$$-\frac{W_\varphi}{r}\frac{\bar\partial(V_\theta r)}{\partial r}+W_z\left(\frac{\bar\partial W_r}{\partial z}-\frac{\bar\partial W_z}{\partial r}\right)+\frac{\bar\partial I}{\partial r}-T\frac{\bar\partial s}{\partial r}-F_r=0 \tag{18}$$

$$\frac{W_r}{r}\frac{\bar\partial(V_\theta r)}{\partial r}+\frac{W_z}{r}\frac{\bar\partial(V_\theta r)}{\partial z}-F_\varphi=0 \tag{19}$$

$$-W_r\left(\frac{\bar\partial W_r}{\partial z}-\frac{\bar\partial W_z}{\partial r}\right)-\frac{W_\varphi}{r}\frac{\bar\partial(V_\theta r)}{\partial z}+\frac{\bar\partial I}{\partial z}-T\frac{\bar\partial s}{\partial z}-F_z=0 \tag{20}$$

where

$$\boldsymbol{F}=-\left(\frac{1}{n_\varphi r\rho}\frac{\partial p}{\partial\varphi}\right)\boldsymbol{n}$$

Equations (11) and (12) become[1]

$$\frac{Ds}{Dt}=W_r\frac{\bar\partial s}{\partial r}+W_z\frac{\bar\partial s}{\partial z}=0 \tag{21}$$

$$\frac{DI}{Dt}=W_r\frac{\bar\partial I}{\partial r}+W_z\frac{\bar\partial I}{\partial z}=0 \tag{22}$$

In addition, there is the orthogonality relation between $\boldsymbol{n}$ and $\boldsymbol{W}$

$$\mu W_r + W_\varphi + \nu W_z = 0 \tag{23}$$

Direct Problem

In the direct problem, values of μ and ν are known from the given shape of S_2, and the $\boldsymbol{F}$ components are related by known values of μ and ν as follows:

$$F_r = \mu F_\varphi \tag{24}$$

$$F_z = \nu F_\varphi \tag{25}$$

The advantage of using I and s as the two primary thermodynamic properties of fluid is clearly seen from the preceding equations. Equations (22) and (21) are considered to represent the energy equation and one equation of motion.[1] Since I and s are invariant along relative streamlines, they are determined over the whole flowfield with due consideration of their initial or inlet values and appropriate change (in s) across the shock. Then Eqs. (18) and (19) are considered to represent the other two equations of motion. Together with the continuity equation (17) and Eqs. (23–25), there are now six independent equations, which can be solved to determine the six dependent variables W_r, W_φ, W_z, F_r, F_φ, and F_z. (For convenience, V_θ is retained in the equations and is related to W_φ by the simple relation $V_\theta = W_\varphi + \omega r$.)

According to the theory of characteristics of system of first-order quasi-linear differential equations with two independent variables,[6] if at least one of the matrices A and B formed by the coefficients of the partial derivatives with respect to r and z, respectively, say, B, is nonsingular, then $|A - \lambda B|$ is the characteristic determinant, $|A - \lambda B| = 0$ is the characteristic equation, and $\lambda = \mathrm{d}r/\mathrm{d}z$ is the slope of the characteristic curve.

Now Eqs. (23–25) do not contain partial derivatives of unknown variables, so that all elements in the last three rows of matrices A and B are equal to zero. These three equations are used to eliminate F_r, F_z, and W_φ. Then, Eqs. (17–19) become

$$\left(1 + \frac{\mu W_r W_\varphi - W_r^2}{a^2}\right)\frac{\partial W_r}{\partial r} + \frac{\mu W_z W_\varphi - W_z W_r}{a^2}\frac{\partial W_r}{\partial z} + \frac{\nu W_r W_\varphi - W_r W_z}{a^2}\frac{\partial W_z}{\partial r} + \left(1 + \frac{\nu W_z W_\varphi - W_z^2}{a^2}\right)\frac{\partial W_z}{\partial z}$$
$$+ \frac{W_r^2 W_\varphi}{a^2}\frac{\partial \mu}{\partial r} + \frac{W_r W_z W_\varphi}{a^2}\left(\frac{\partial \mu}{\partial z} + \frac{\partial \nu}{\partial r}\right) + \frac{W_z^2 W_\varphi}{a^2}\frac{\partial \nu}{\partial z} + \frac{\omega^2 r W_r}{a^2} + W_r\frac{\partial \ln \tau}{\partial r} + W_z\frac{\partial \ln \tau}{\partial z} = 0 \tag{26}$$

$$\mu W_\varphi \frac{\partial W_r}{\partial r} + W_z\frac{\partial W_r}{\partial z} + (\nu W_\varphi - W_z)\frac{\partial W_z}{\partial r} + W_\varphi W_r \frac{\partial \mu}{\partial r} + W_\varphi W_z \frac{\partial \nu}{\partial r} - \frac{W_\varphi^2}{r} - 2\omega W_\varphi + \frac{\partial I}{\partial r} - T\frac{\partial s}{\partial r} - \mu F_\varphi = 0 \tag{27}$$

$$\mu W_r\frac{\partial W_r}{\partial r} + \mu W_z\frac{\partial W_r}{\partial z} + \nu W_r\frac{\partial W_z}{\partial r} + \nu W_z\frac{\partial W_z}{\partial z} + W_r^2\frac{\partial \mu}{\partial r} + W_z W_r\frac{\partial \mu}{\partial z} + W_r W_z\frac{\partial \nu}{\partial r} + W_z^2\frac{\partial \nu}{\partial z} + F_\varphi = 0 \tag{28}$$

The partial derivatives of F_φ do not appear in these three equations. Substituting F_φ from Eq. (28) into Eq. (27) results in

$$-\mu\nu\frac{\partial W_r}{\partial r} + (1+\mu^2)\frac{\partial W_r}{\partial z} - (1+\nu^2)\frac{\partial W_z}{\partial r} + \mu\nu\frac{\partial W_z}{\partial z} - \nu W_r\frac{\partial \mu}{\partial r} + \mu W_r\frac{\partial \mu}{\partial z} - \nu W_z\frac{\partial \nu}{\partial r} + \mu W_z\frac{\partial \nu}{\partial z} - \frac{W_\varphi^2}{W_z r}$$
$$-2\omega\frac{W_\varphi}{W_z} + \frac{1}{W_z}\frac{\partial I}{\partial r} - \frac{T}{W_z}\frac{\partial s}{\partial r} = 0 \tag{29}$$

Finally, the characteristic matrix $A - \lambda B$ formed by the coefficients of partial derivatives in Eqs. (26) and (29) is as shown below:

$$\begin{bmatrix} 1 + \dfrac{\mu W_r W_\varphi - W_r^2}{a^2} - \dfrac{\mu W_z W_\varphi - W_z W_r}{a^2} & \dfrac{\nu W_r W_\varphi - W_r W_z}{a^2} - \left(1 + \dfrac{\nu W_z W_\varphi - W_z^2}{a^2}\right)\lambda \\ -\mu\nu - (1+\mu^2)\lambda & -(1+\nu^2) - \mu\nu\lambda \end{bmatrix}$$

Equating the characteristic determinant equal to zero gives

$$\left[(1+\mu^2) - (1+\mu^2+\nu^2)\frac{W_z^2}{a^2}\right]\lambda^2 + 2\left[\mu\nu + (1+\mu^2+\nu^2)\frac{W_r W_z}{a^2}\right]\lambda + \left[(1+\nu^2) - (1+\mu^2+\nu^2)\frac{W_r^2}{a^2}\right] = 0 \tag{30}$$

from which

$$\lambda = \frac{-a^2\mu\nu - (1+\mu^2+\nu^2)W_r W_z \pm \sqrt{(1+\mu^2+\nu^2)(W^2 - a^2)}}{a^2(1+\mu^2) - (1+\mu^2+\nu^2)W_z^2}$$

Hence, at all points on the S_2 surface where $W > a$, there are two distinct real roots and the system of the differential equations is hyperbolic.

Inverse Problem

In the inverse problem of fluid flow on an S_2 surface, usually the angular momentum $V_\theta r$ (or V_θ, or W_φ) and τ are prescribed by the designer as functions of (r, z). In order to obtain a smooth integral S_2 surface, the following integrability condition must be satisfied:[1]

$$\frac{\partial}{\partial r}\left(\frac{\nu}{r}\right) = \frac{\partial}{\partial z}\left(\frac{\mu}{r}\right)$$

or

$$\frac{\partial \nu}{\partial r} - \frac{\nu}{r} - \frac{\partial \mu}{\partial z} = 0 \tag{31}$$

Similarly to the direct problem, if it is considered that I and s are determined by Eqs. (21) and (22), there are five remaining equations, (17–19), (23), and (31), to determine the remaining five dependent variables, W_r, W_z, μ, ν, and F_φ. (F_r and F_z are given by μF_φ and νF_φ, respectively).

Since Eqs. (19) and (23) do not contain partial derivatives of the unknown variables, they are used first to eliminate F_φ and ν from the equations. Substituting Eq. (19) into Eq. (18)

results in

$$\frac{-\partial W_z}{\partial r}+\frac{\nu}{r}\frac{\partial(V_\theta r)}{\partial r}-\frac{\mu}{r}\frac{\partial(V_\theta r)}{\partial z}+\frac{\partial W_r}{\partial z}+\frac{1}{W_z}\frac{\partial I}{\partial r}-\frac{T}{W_z}\frac{\partial s}{\partial r}=0 \tag{32}$$

Substituting ν from Eq. (23) into Eq. (31) results in

$$\mu\frac{\partial W_r}{\partial r}+\nu\frac{\partial W_z}{\partial r}+W_r\frac{\partial\mu}{\partial r}+W_z\frac{\partial\mu}{\partial z}+\frac{\partial W_\varphi}{\partial r}+\frac{\nu W_z}{r}=0 \tag{33}$$

Now Eqs. (17), (32), and (33) are a system of three first-order differential equations for three dependent variables W_r, W_z, and μ. The two matrices A and B formed, respectively, by the coefficients of the partial derivatives with respect to r and z are nonsingular. The characteristic matrix of the system of equations $A-\lambda B$ is as follows:

$$\begin{bmatrix}\left(1-\frac{W_r^2}{a^2}\right)+\frac{W_rW_z}{a^2}\lambda & -\frac{W_rW_z}{a^2}-\left(1-\frac{W_z^2}{a^2}\right)\lambda & 0\\ -\lambda & -1 & 0\\ \mu & \nu & W_r-W_z\lambda\end{bmatrix}$$

This is a lower triangular matrix with submatrix blocks (one is the four upper-left elements, and the other is the lower-right element) as the diagonal elements. With the aid of Laplace's theorem, the determinant of this characteristic matrix can be factorized into the subdeterminants of these diagonal elements. Equating these subdeterminants equal to zero gives

$$\lambda=\frac{dr}{dz}=\frac{W_r}{W_z} \tag{34}$$

and

$$\left(1-\frac{W_z^2}{a^2}\right)\lambda^2+2\frac{W_rW_z}{a^2}\lambda+\left(1-\frac{W_r^2}{a^2}\right)=0 \tag{35}$$

from which

$$\lambda=\frac{-W_rW_z\pm\sqrt{a^2\left(W_r^2+W_z^2-a^2\right)}}{a^2-W_z^2}$$

Equation (34) means that there is a stream characteristic. Equation (35) has two distinct real roots, i.e., the system of partial differential equations is hyperbolic, at all points on the S_2 surface where $(W_r^2+W_z^2)>a^2$, or $W_1>a$.

Determining the Type of the Second-Order Principal Equation Governing S_2 Flow

The actual solution of either the direct or inverse problem discussed in the preceding section is carried out with the use of a stream function, which is defined on the stream surface[1] on the basis of the continuity equation (16):

$$\frac{\partial\psi}{\partial r}=\tau\rho W_z \tag{36}$$

$$\frac{\partial\psi}{\partial z}=-\tau\rho W_r \tag{37}$$

Direct Problem

By the use of Eqs. (5–9), (36), and (37), Eq. (29) is transformed into

$$\left[(1+\nu^2)-(1+\mu^2+\nu^2)\frac{W_r^2}{a^2}\right]\frac{\partial^2\psi}{\partial r^2}-2\left[\mu\nu+(1+\mu^2+\nu^2)\frac{W_rW_z}{a^2}\right]\frac{\partial^2\psi}{\partial r\,\partial z}+\left[(1+\mu^2)-(1+\mu^2+\nu^2)\frac{W_z^2}{a^2}\right]\frac{\partial^2\psi}{\partial z^2}+N\frac{\partial\psi}{\partial r}+M\frac{\partial\psi}{\partial z}=0 \tag{38}$$

where

$$N=-(1+\nu^2)\times\left[\frac{\partial\ell n\,\tau}{\partial r}-\frac{1}{R}\frac{\partial s}{\partial r}+\frac{1}{a^2}\left(\frac{\partial I}{\partial r}+\omega^2r+W_rW_\varphi\frac{\partial\mu}{\partial r}+W_\varphi W_z\frac{\partial\nu}{\partial r}\right)\right]+\mu\nu\left[\frac{\partial\ell n\,\tau}{\partial z}-\frac{1}{R}\frac{\partial s}{\partial z}+\frac{1}{a^2}\left(\frac{\partial I}{\partial z}+W_rW_\varphi\frac{\partial\mu}{\partial z}+W_\varphi W_z\frac{\partial\nu}{\partial z}\right)\right]+\frac{a^2-W^2}{a^2}\times\left[\nu\frac{\partial\nu}{\partial r}-\mu\frac{\partial\nu}{\partial z}-\frac{\nu W_\varphi}{rW_z}-\frac{2\omega\nu}{W_z}-\frac{1}{W_z^2}\left(\frac{\partial I}{\partial r}-T\frac{\partial s}{\partial r}\right)\right]$$

$$M=\mu\nu\left[\frac{\partial\ell n\,\tau}{\partial r}-\frac{1}{R}\frac{\partial s}{\partial r}+\frac{1}{a^2}\left(\frac{\partial I}{\partial r}+\omega^2r+W_rW_\varphi\frac{\partial\mu}{\partial r}+W_\varphi W_z\frac{\partial\nu}{\partial r}\right)\right]-(1+\mu^2)\times\left[\frac{\partial\ell n\,\tau}{\partial z}-\frac{1}{R}\frac{\partial s}{\partial z}+\frac{1}{a^2}\left(\frac{\partial I}{\partial z}+W_rW_\varphi\frac{\partial\mu}{\partial z}+W_\varphi W_z\frac{\partial\nu}{\partial z}\right)\right]-\frac{a^2-W^2}{a^2}\left(\nu\frac{\partial\mu}{\partial r}-\mu\frac{\partial\mu}{\partial z}\right)$$

It is seen from the coefficients of the second-order partial derivatives that when $W>a$ or $<a$, Eq. (38) is hyperbolic or elliptic. This is in agreement with the result obtained in the preceding section.

Inverse Problem

Consider again the design problem in which $V_\theta r=f(r,z)$ is prescribed by the designer. By the use of Eqs. (5–9), (36), and (37), Eq. (32) can be put into the following form:

$$\left(1-\frac{W_r^2}{a^2}\right)\frac{\partial^2\psi}{\partial r^2}-2\frac{W_rW_z}{a^2}\frac{\partial^2\psi}{\partial r\,\partial z}+\left(1-\frac{W_z^2}{a^2}\right)\frac{\partial^2\psi}{\partial z^2}+N\frac{\partial\psi}{\partial r}+M\frac{\partial\psi}{\partial z}=0 \tag{39}$$

where

$$N=-\frac{\partial\ell n\,\tau}{\partial r}+\frac{1}{R}\frac{\partial s}{\partial r}-\frac{1}{a^2}\left(\frac{\partial I}{\partial r}-W_\varphi\frac{\partial W_\varphi}{\partial r}+\omega^2r\right)+\frac{a^2-(W_r^2+W_z^2)}{a^2W_z^2}\left[-\frac{\partial I}{\partial r}+T\frac{\partial s}{\partial r}-\frac{\nu}{r}\frac{\partial(V_\theta r)}{\partial r}+\frac{\mu}{r}\frac{\partial(V_\theta r)}{\partial z}\right]$$

$$M=-\frac{\partial\ell n\,\tau}{\partial z}+\frac{1}{R}\frac{\partial s}{\partial z}-\frac{1}{a^2}\left(\frac{\partial I}{\partial z}-W_\varphi\frac{\partial W_\varphi}{\partial z}\right)$$

From the coefficients of the second-order partial derivatives, this principal equation is seen to be hyperbolic or elliptic when the meridional velocity $W_1 = (W_r^2 + W_z^2)^{\frac{1}{2}} \gtrless$ the speed of sound, respectively. This is again in agreement with the result obtained in the preceding section.

It should be noted that in the preceding section the type of the system of equations is not determined by the whole system of equations. In the direct problem, the two wave characteristics are actually determined by Eqs. (26) and (29), which are the continuity equation and the equation of motion in the radial direction. Here, these two equations are combined to form the principal equation (38). Also, in the inverse problem of the preceding section, the two wave characteristics are finally determined by Eqs. (17) and (32), which are again the continuity equation and the equation of motion in the radial direction, respectively. Here, in this section, these two equations are combined to form the principal equation (39). Therefore, it is only natural that exactly the same results are obtained by the consideration of either the set of first-order differential equations or the single principal equation of second-order. The type of the principal equation is consistent with that of the system of first-order equations.

Basis of an Accurate Solution of an Inverse Problem of Transonic S_2 Flow Using an Elliptic Solver

In an inverse problem of transonic flow on an S_2 stream surface (with $V_\theta r$ prescribed), if the meridional velocity component does not exceed the speed of sound, the nature of the problem is the same in the supersonic region as in the subsonic region. Therefore, an elliptic algorithm can be used for the whole flowfield. This avoids the trouble of dealing with the mixed type of equation, as in the direct problem, and greatly simplifies the computation.

This may appear to be bewildering, but there are similar two-dimensional aerodynamic examples. Flow past a swept cylindrical wing of infinite span is a good external flow example. In this case, the equation has been shown to be elliptic, provided the normal (to spanwise) component of the flow velocity is subsonic (cf. Ref. 8, e.g.). We can, in addition, cite the axisymmetric flowfield, in which the equation is elliptic when the meridional velocity is subsonic, and so on. Obviously, the criterion that determines the type of these differential equations varies with the condition that renders the flow two-dimensional.

The existence of a type-distinguishing criterion that does not obey $W \gtrless a$, does not violate the three rules on the propagation of a disturbance signal in supersonic flow. In steady three-dimensional flow, the small-disturbance wave emitted at each time by a point source is an enlarging spherical surface. In the supersonic case, all the spherical surfaces possess an envelope surface, which is the conical surface of the Mach cone, and the disturbance is restricted to the interior of the Mach cone and concentrated in the neighborhood of the conical surface. In the subsonic case, however, this envelope does not exist. The flow past an infinite-span cylindrical wing swept in the x-y plane is taken as an example of a two-dimensional problem, in which the variation of the flow parameters in the spanwise direction is considered to be zero (Fig. 4). The simplest element of disturbance source (point source) in this two-dimensional problem is in fact an infinite straight line in the direction of the span L. When the flow is supersonic and steady, each point of the line source emits a Mach cone with a Mach angle α. If the normal velocity component is supersonic, i.e., $M \sin\beta > 1$ or $\beta > \alpha$, L is spacelike, and all the Mach cones possess an envelope, which is composed of the two characteristic surfaces or, physically, the Mach waves. The disturbance is restricted to the interior of the Mach waves and concentrated in the neighborhood of the Mach waves (Fig. 4a). If $M \sin\beta < 1$ or $\beta < \alpha$, however, L is timelike and the Mach cones spread over the whole space with no envelope. In this case, there is no restriction on the propagation of the disturbance, and no characteristic surface or Mach wave exists (Fig. 4b). Consequently, the type of the equation conforms to the physical phenomenon.

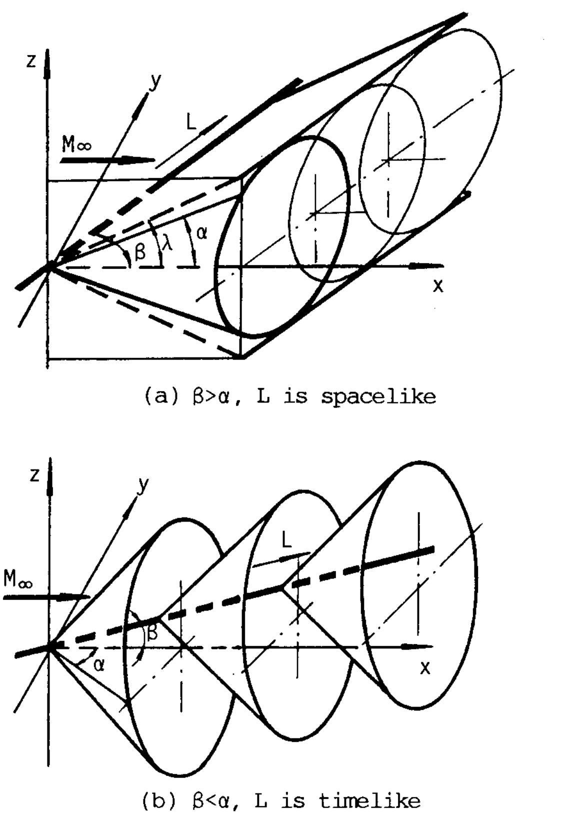

Fig. 4 Mach cones emitted from the line disturbance source in direction L in a two-dimensional problem.

Let us place a test source of a small disturbance at a certain point on the S_2 surface and review the propagation of the disturbance. Since the value of $V_\theta r$ is prescribed in the inverse problem, it should not vary with boundary conditions and with solution; therefore, $V_\theta r$ is a quantity that is not affected by the disturbance. This requirement determines that the test source has to be axially symmetric; that is, it is a circular ring. Provided that in the supersonic flow region the meridional velocity is subsonic, the circular ring is timelike, and the Mach cones do not possess an envelope, and have no characteristic surface—Mach wave forms. Thus, the type of equation is the same as that in the subsonic flow region.

It is necessary to consider another question: in turbomachine transonic flow, shock waves might exist, even though $W_1 < a$. As is well known, when an equation of mixed type is used to describe such flow, the solution possesses discontinuities. It is also known from mathematics that the solution of an elliptic equation is smooth, provided the coefficients are smooth. However, the solution of an equation of any type will have discontinuities if the coefficients have discontinuities. That any one of the shock wave surfaces is parallel to $V_\theta r$ (or W_φ) is contrary to the condition $W_1 < a$, because the normal (to shock surface) velocity component of the flow in front of the shock must be larger than the speed of sound. Thus, the shock surfaces are not parallel to $V_\theta r$, and the value of $V_\theta r$ must have an abrupt jump across each shock surface corresponding to the jump of the normal velocity component. In the inverse problem of S_2 with $V_\theta r$ prescribed, the value of $V_\theta r$ should possess an abrupt change across each shock, and the magnitude of the jump should correspond to the strength and orientation of the shock. This will force the solution to have an appropriate jump across the shock. The solution that satisfies the shock relation at the discontinuity is the correct solution of the inverse problem of transonic flow on an S_2 surface.

Numerical Calculations

Practical calculation of the inverse problem of transonic flow on S_2 surface can be carried out by exploiting available computer codes originally programmed for the inverse problem of subsonic flow on S_2 and $V_\theta r$ prescribed, such as the one reported in Ref. 9. The principal equation in the stream function employing general nonorthogonal curvilinear coordinates and corresponding nonorthogonal velocity components is[10]

$$\frac{\sqrt{a_{11}}}{\rho\tau\sin\theta_{12}\sqrt{a_{22}}}\frac{\partial^2\psi}{\partial(x^2)^2} - 2\frac{\cos\theta_{12}}{\rho\tau\sin\theta_{12}}\frac{\partial^2\psi}{\partial x^1\partial x^2} + \frac{\sqrt{a_{22}}}{\rho\tau\sin\theta_{12}\sqrt{a_{11}}}\cdot\frac{\partial^2\psi}{\partial(x^1)^2}$$
$$+\left[\partial\left(\frac{\sqrt{a_{11}}}{\rho\tau\sin\theta_{12}\sqrt{a_{22}}}\right)\Big/\partial x^2 - \partial\left(\frac{\cos\theta_{12}}{\rho\tau\sin\theta_{12}}\right)\Big/\partial x^1\right]\frac{\partial\psi}{\partial x^2}$$
$$+\left[\partial\left(\frac{\sqrt{a_{22}}}{\rho\tau\sin\theta_{12}\sqrt{a_{11}}}\right)\Big/\partial x^1 - \partial\left(\frac{\cos\theta_{12}}{\rho\tau\sin\theta_{12}}\right)\Big/\partial x^2\right]\frac{\partial\psi}{\partial x^1}$$
$$= A + F \tag{40}$$

where

$$A = \frac{\sqrt{a_{11}}}{W^1}\left(-\frac{W_\varphi}{r}\frac{\partial(V_\theta r)}{\partial x^2} + \frac{\partial I}{\partial x^2} - T\frac{\partial s}{\partial x^2}\right)$$

$$F = \left(-\frac{n_2}{n_\varphi}\right)\left[\frac{\partial(V_\theta r)}{\partial x^1} + \frac{\sqrt{a_{11}}}{\sqrt{a_{22}}}\frac{W^2}{W^1}\frac{\partial(V_\theta r)}{\partial x^2}\right]\frac{\partial\varphi}{\partial x^2}$$

Here the x^1 and x^2 coordinates lie on the meridional plane and $x^3 = \varphi$. Equation (40) is elliptic. The difference equation is obtained by the use of the central-difference formula, and the resulting equations are solved by the method of successive line overrelaxation. An appropriate radial variation of entropy may be prescribed to account for the effect of the radial variation of the isentropic efficiency of a rotor or the pressure recovery in a stator.[9] Their values are usually taken by experience when no accurate ones can be used, e.g., at the beginning of design.

To demonstrate the present method, the $S_{2,m}$ flow in the transonic rotor of Ref. 11 has been calculated. The rotor was designed by DFVLR[11] for a total pressure ratio of 1.51. The relative tip Mach number reaches 1.4, but the meridional velocity component is subsonic everywhere. Measured data of this rotor have been published.[12] From the available Mach number contour plots on the S_1 surfaces located at a number of spanwise positions for peak efficiency operating condition at 100% design speed,[12] the projection on the meridional plane of the Mach number contours on the $S_{2,m}$ surface is constructed and given in Fig. 5.

The computational mesh in our calculation has 20 axial points and 11 radial points within the blade passage. The increases of $V_\theta r$ and s from inlet to exit are determined from measured data.[12] However, due to the limited data available, the variation of the thickness of stream sheet τ is approximately taken to be proportional to the channel width.

In subsonic compressors, $V_\theta r$ and s are usually assumed to increase in a nearly linear manner across a blade row. When the $V_\theta r$ and s prescribed are of this type, (referred to as calculation I) the flowfield obtained is shown in Fig. 6. It can be seen that, although there is a supersonic region, the flowfield has no shock discontinuity and is quite different from that shown in Fig. 5. The reason is that, in a transonic rotor, $V_\theta r$ and s actually possess abrupt changes across the shock and are quite different from those of the subsonic rotor.

When the distributions of $V_\theta r$ and s (Fig. 7), which are similar to the real ones in the transonic rotor are prescribed (referred to as calculation II) (with the same inlet and exit values as those in calculation I), the result of the calculation obtained by using the same elliptic algorithm shows clearly a shock in the tip region and the Mach number contours obtained (Fig. 8) are quite similar to those shown in Fig. 5. The chordwise variations of pressure along three streamlines on the $S_{2,m}$ surface are shown in Fig. 9. They also show clearly the abrupt changes across the shock on the upper two streamlines. The coordinate of the $S_{2,m}$ surface, $\varphi = \varphi(r,z)$, obtained in the calculation (i.e., the shape of the $S_{2,m}$ surface is obtained at the end of the calculation in an inverse problem) multiplied by the mean radius r_m, is shown along three streamlines in Fig. 10.

Although the strict shock relations are not yet used in the solution and the discontinuities of $V_\theta r$ and s are smoothed out slightly, such calculation is of practical value. It is similar to the calculation in the direct problem of stream surface flow in which the strict shock relations are not used and the shock is not dealt with exactly in the field computation—the so-called shock-capturing method.

If it is desired to have a more precise solution of transonic $S_{2,m}$ flow, the following connecting relations instead of the

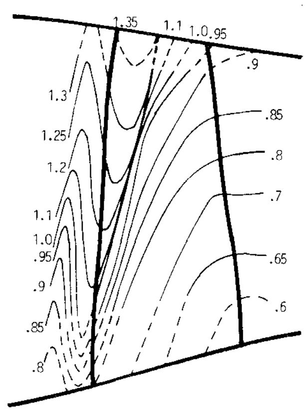

Fig. 5 Projection on meridional plane of Mach number contours on $S_{2,m}$.

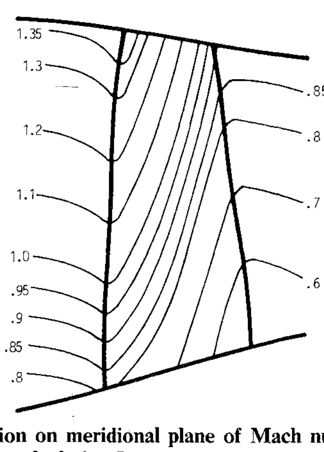

Fig. 6 Projection on meridional plane of Mach number contours on $S_{2,m}$ obtained in calculation I.

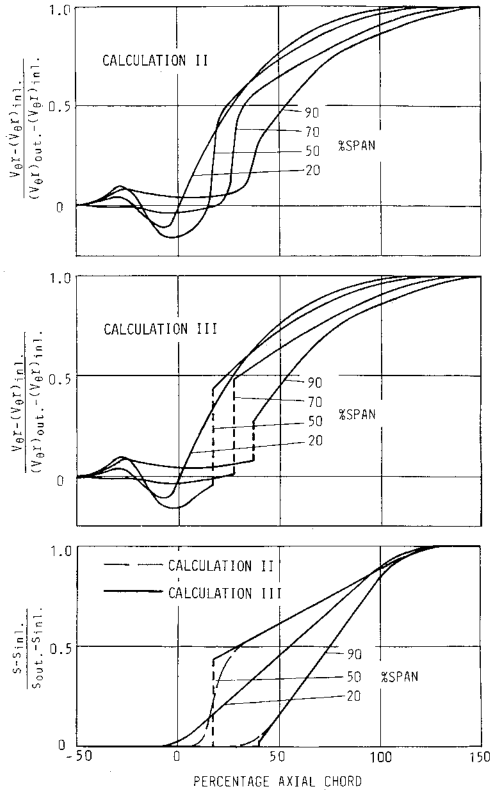

Fig. 7 $V_\theta r$ and s prescribed in calculations.

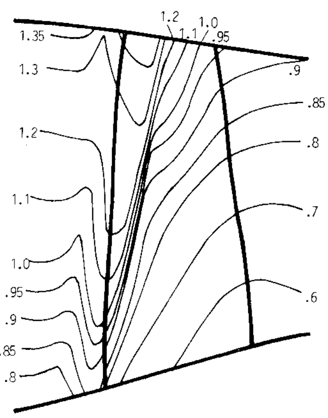

Fig. 8 Projection on meridional plane of Mach number contours on $S_{2,m}$ obtained in calculation II.

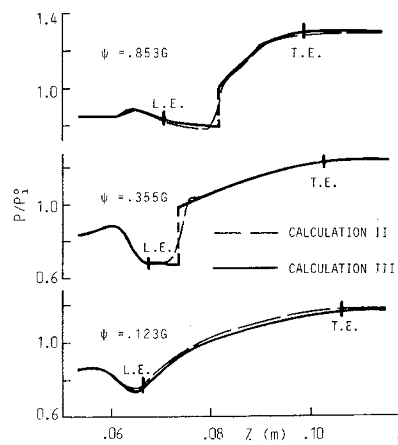

Fig. 9 Calculated chordwise variation of pressure on $S_{2,m}$.

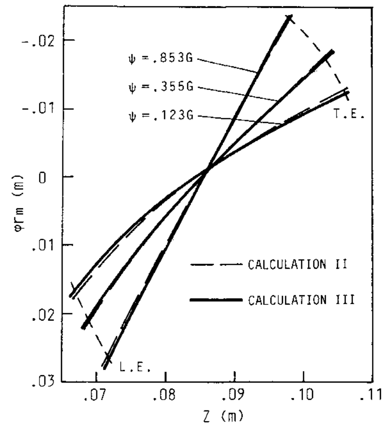

Fig. 10 Calculated chordwise variation of φr_m of $S_{2,m}$ surface.

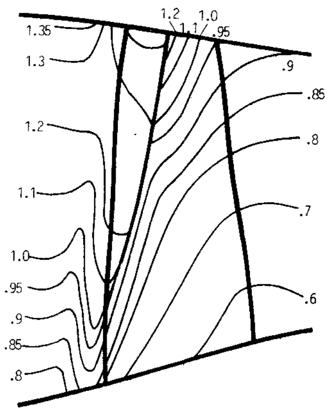

Fig. 11 Projection on meridional plane of Mach number contours on $S_{2,m}$ obtained in calculation III.

original differential equations should be employed at the shock discontinuity.

Continuity equation:

$$[\psi]=0 \tag{41}$$

Equation of motion:

$$[W_t]=0 \tag{42}$$

Energy equation:

$$[I]=0 \tag{43}$$

Entropy relation:

$$\left[\frac{s}{R}\right]=\frac{\kappa}{\kappa-1}\ell n\left[\frac{2}{(\kappa+1)M_{1n}^2}+\frac{\kappa-1}{\kappa+1}\right]+\frac{1}{\kappa-1}\ell n\left[\frac{2\kappa}{\kappa+1}M_{1n}^2-\frac{\kappa-1}{\kappa+1}\right] \tag{44}$$

Condition of continuity of stream surface:

$$[\varphi]=0 \tag{45}$$

where $\varphi=\varphi(r,z)$ is the coordinate of S_2 surface in the circumferential direction. This corresponds to the method of "separate-region calculation with shock fitting" in the direct problems of S_1 stream surface flow.[13,14] A code that is the result of embedding the preceding relations into the subsonic code of Ref. 9 has been programmed (the details of which will

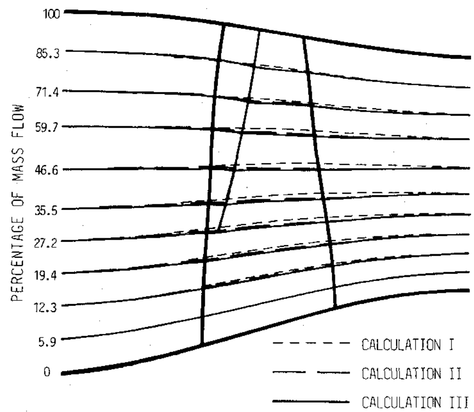

Fig. 12 Difference between streamlines obtained in calculations.

be published in another paper). By putting in the distribution of $V_\theta r$ with a finite jump (Fig. 7), a solution including a genuine shock discontinuity is obtained (referred to as calculation III). Some results are given in Figs. 9–11. They are only slightly different from those obtained in calculation II.

Figure 12 shows the difference between the meridional projection of the streamlines on $S_{2,m}$ surface obtained in the calculations. Because, in the iterative calculation between S_1 and S_2 to obtain a three-dimensional solution, the shapes of S_1 surfaces and the thickness of S_1 stream filaments are given by these streamlines, the inaccuracy in the solution of the S_2 inverse problem would affect the accuracy of the three-dimensional solution. The accurate prescribed distributions of $V_\theta r$ and τ can be obtained only from the S_1 calculations; that is, iterative solution between S_1 and S_2 is necessary. Both calculations II and III converge without any difficulty. It is believed that the present method is still practical for transonic flows having higher Mach number as long as the meridional velocity is subsonic.

Conclusion

The type of differential equations governing the fluid flow on the S_2 surface in a turbomachine is rigorously analyzed in detail on the basis of a system of first-order differential equations and also on the basis of the second-order principal equation in stream function. Consistent results are obtained for the direct and inverse problems. In the direct problem, where the shape of the S_2 surface is given, the equation is hyperbolic or elliptic when the relative velocity is supersonic or subsonic. In the inverse problem, where the variation of $V_\theta r$ is prescribed by the designer, the equation is hyperbolic or elliptic when the meridional velocity component is greater or less than the local speed of sound.

Illustrative examples verify the facts that, when the variations of $V_\theta r$ and s prescribed in the inverse problem of transonic S_2 surface flow do not possess abrupt changes, the solution yields no shock discontinuity, and when the prescribed values of $V_\theta r$ and s do have abrupt changes, the solution yields a clear shock. The latter provides a simple accurate solution of the transonic flow on S_2 surfaces in turbomachines. It may be used, in particular, as a follow-up calculation to the calculation procedure commonly used in transonic compressor design and, in general, in the calculation of three-dimensional transonic flow in turbomachines employing the method of iterative calculation of S_1 and S_2 stream surface flows.

References

[1]Wu, C.H., "A General Theory of Three-Dimensional Flow in Subsonic and Supersonic Turbomachines of Axial, Radial, and Mixed-Flow Types," ASME Paper 50-A-79 (ASME Transactions, Nov. 1952); also NACA TN-2604, 1952.

[2]Academia Sinica and Shenyang Aeroengine Co., "Theory, Method and Application of Three-Dimensional Flow Design of Transonic Axial-Flow Compressors," Res. Rept. 1976, Chinese Academy of Sciences, or *Journal of Engineering Thermophysics*, Vol. 1, No. 1, Feb. 1980, pp. 44–54 (in Chinese).

[3]Seyler, D.R. and Smith, L.H. Jr., "Single Stage Experimental Evaluation of High Mach Number Compressor Rotor Blading, Pt. I-Design of Rotor Blading," NASA CR-54581, Appendix B, 1967.

[4]Monsarrat, N.T., Keenan, K.J., and Tramm, P.C., "Design Report, Single-Stage Evaluation of Highly-Loaded High-Mach-Number Compressor Stages," NASA CR-22562, Appendix I, 1969.

[5]Janetzke, D.C., Ball, C.L., and Hager, R.D., "Performance of 1380-Foot-Per-Second-Tip-Speed Axial-Flow Compressor Rotor with Blade Tip Solidity of 1.1," NASA TM X-2449, 1972.

[6]Courant, R. and Hilbert, D., *Methods of Mathematical Physics*, Vol. II, Interscience, New York, 1962, Chap. III.

[7]Wu, C.H., Discussion to a paper entitled "A Practical Solution of a Three-Dimensional Flow Problem of Axial-Flow Turbomachinery," *ASME Transactions*, July, 1953, pp. 802–803.

[8]Oswatitsch, K., *Gas Dynamics*, Academic Press, Orlando, FL, Chap. 6, 1956.

[9]Zhu, R.G., "Flow-Field Line-Relaxation Solution for Inverse Problem of Flow Along S_2 Relative Stream Surface Employing Non-Orthogonal Curvilinear Coordinates and Corresponding Non-Orthogonal Velocity Components," *Journal of Engineering Thermophysics*, Vol. 1, No. 1, Feb. 1980, pp. 28–35 (in Chinese).

[10]Wu, C.-H., "Three-Dimensional Turbomachine Flow Equations Expressed with Respect to Non-Orthogonal Curvilinear Coordinates and Methods of Solution," Lecture Notes, China University of Science and Technology, 1975; also *Proceedings of 3rd ISABE*, 1976, pp. 233–252.

[11]Strinning, P.E. and Dunker, R.J., "Aerodynamische und Schanfelauslegung einer Transsonischen Axialverdichterstufe," *Forschungsberichte Verbrennungs-Kraftmashinen*, DFVLR, Vol. 178, 1975.

[12]McDonald, P.W. et al., "A Comparison Between Measured and Computed Flow Fields in a Transonic Compressor Rotor," *Journal of Engineering for Power*, Vol. 102, Oct. 1980, pp. 883–890.

[13]Wu, C.H., Wu, W., Hua, Y., and Wang, B., "Transonic Cascade Flow with Given Shock Shape Solved by Separate Subsonic and Supersonic Computations," *Symposium on Computational Methods in Turbomachinery*, I. Mech. Engg., England, 1984, pp. 133–140.

[14]Wu, W., Wu, C.-H., and Yu, D., "Transonic Cascade-Flow Solved by Separate Supersonic and Subsonic Computations with Shock Fitting," *Journal of Engineering for Power and Gas Turbine*, Vol. 107, April 1985, pp. 329–336.